DOI:10. 11714/acta. snus. ZR20250139

基于奇偶性的四元数线性正则变换分解及仿真*

姬锦仪[1]，冯子健[2]，郭子园[1]，杨茜[1]，冯强[1,3]

1. 延安大学数学与计算机科学学院，陕西 延安 716000
2. 天津大学数学学院，天津 300354
3. 陕西省能源大数据智能处理省市共建重点实验，陕西 延安 716000

摘 要：在信号处理与代数分析领域，四元数及其变换理论因其独特的数学特性受到广泛关注. 本文研究右侧四元数线性正则变换（QLCT）. 首先，介绍了四元代数的基本概念，包括四元数的表示、运算规则、共轭、范数等，并给出了QLCT的定义；其次，基于信号的奇偶性，提出了右侧QLCT的偶分量与奇分量分解方法，深入探讨了该变换的相关性质并给出证明. 最后，以高斯信号为例进行仿真实验，验证了理论推导，并展示了四元数值信号及其QLCT变换后各分量的特性.

关键词：四元数；四元数线性正则变换；高斯信号

中图分类号：TN911.72 **文献标志码**：A **文章编号**：2097－0137（XXXX）XX－0001－09

Decomposition and simulation of quaternion linear canonical transform based on parity properties

JI Jinyi[1], *FENG Zijian*[2], *GUO Ziyuan*[1], *YANG Xi*[1], *FENG Qiang*[1,3]

1. School of Mathematics and Computer Science, Yan'an university, Yan'an 716000, China
2. School of Mathematics, Tianjin University, Tianjin 300354, China
3. Key Experimental Project Co-conducted by Province and City for Intelligent Processing of Energy Big Data in Shaanxi Province, Yan'an 716000, China

Abstract: In the fields of signal processing and algebraic analysis, quaternions and their transformation theory have attracted extensive attention due to their unique mathematical properties. This paper studies the right-sided quaternion linear canonical transform (QLCT). Firstly, it introduces the basic concepts of quaternion algebra, including the representation of quaternions, operation rules, conjugation, and norm; and provides the definition of QLCT. Secondly, based on the parity of the signal, it proposes the decomposition method of even and odd components of the QLCT, deeply explores the relevant properties of this transformation, and provides proofs. Finally, taking gaussian signals as an example, simulation experiments are conducted to verify the theoretical derivation and demonstrate the characteristics of each component of the quaternion-valued signal and its QLCT transformation.

Key words: quaternions; quaternion linear canonical transform; gaussian signal

* **收稿日期**：2025－07－24 **录用日期**：2025－10－27 **网络首发日期**：XXXX－XX－XX

基金项目：国家自然科学基金（62261055，61861044）；
陕西省自然科学基金（2022JM-400，2023-JC-YB-085）；
2024年国家级大学生创新创业训练计划项目（2024101719039）

作者简介：姬锦仪（2004年生），男；**研究方向**：分数域信号处理理论与方法；E-mail：1634979926@qq.com

通信作者：冯强（1975年生），男；**研究方向**：傅里叶分析及其应用、分数域信号处理理论与方法；
E-mail：yadxfq@yau.edu.cn

全文阅读
ZR20250139

在现代信号处理、图像处理以及多维数据分析领域，四元数(Quaternion)(Hamilton, 1866)凭借其能够高效描述三维旋转和处理矢量数据的特性而备受关注. 作为复数的扩展，四元数构成的非交换代数结构为处理具有多分量耦合特性的复杂数据提供了独特的数学工具，与此同时，线性正则变换(LCT，Linear Canonical Transform)(Healy et al., 2016)作为傅里叶变换(Fourier Transform)(Bracewell et al., 1989)的广义形式，凭借其灵活的参数调节能力，在时频分析、滤波等领域(Shah et al., 2022; Liu et al., 2025)展现出显著优势. 将LCT推广至四元数域形成的四元数线性正则变换(QLCT，Quaternion Linear Canonical Transform)(Gao et al., 2020; Zhu et al., 2021; Bhat et al., 2022; Hu et al., 2023; Siddiqui et al., 2024)，为复杂数据的时频分析与处理开辟了新的研究方向.

四元数的研究可追溯至其作为复数扩展的提出，随着研究深入，学者们围绕四元数代数及其在信号处理中的应用已取得了一系列重要成果(Li et al., 2020; Xiang et al., 2023; Mei et al., 2023; Majorkowaka-Mech et al., 2023; Wang et al., 2024a, 2024b, 2024c; Yang et al., 2024, 2025; Jiang et al., 2025). 近年来，线性正则变换的优良特性进一步推动了其在四元数域上的推广，初步建立了四元数线性正则变换的基本框架，相关研究主要集中在变换定义、基本性质及初步应用探索等方面，为深入理解QLCT奠定了基础.

尽管QLCT已受到关注，但针对右侧QLCT的系统分析仍有待深化，特别是在基于奇偶性的分量分解方法、关键性质的推导与验证等方面，尚未形成系统化的理论成果. 因此，本文旨在深入研究右侧QLCT的特性，提出并严格推导基于奇偶性的右侧QLCT分量分解方法，建立并证明该变换的核心性质及相关推论，通过严谨的数学推导，完善右侧QLCT的理论体系，为其在复杂信号处理中的实际应用提供更坚实的理论支撑.

1 预备知识

1.1 四元代数

四元数是复数的扩展，构成一个在实数域$\mathbb{R}$上的结合但非交换代数. 四元数的集合用$\mathbb{H}$表示. 任何四元数$q \in \mathbb{H}$都可以表示为以下形式：

$$q = q_r + \mathrm{i}q_\mathrm{i} + \mathrm{j}q_\mathrm{j} + \mathrm{k}q_\mathrm{k},$$

其中$q_r, q_\mathrm{i}, q_\mathrm{j}, q_\mathrm{k} \in \mathbb{R}$，$\mathrm{i}q_\mathrm{i} + \mathrm{j}q_\mathrm{j} + \mathrm{k}q_\mathrm{k}$表示矢量，记作$\vec{q}$.

四元数的乘法规则如下：

$$\mathrm{ij} = -\mathrm{ji} = \mathrm{k}, \quad \mathrm{jk} = -\mathrm{kj} = \mathrm{i}, \quad \mathrm{ki} = -\mathrm{ik} = \mathrm{j}, \quad \mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1.$$

对于两个四元数$p, q \in \mathbb{H}$，分别具有矢量部分$\vec{p}, \vec{q}$，则两个四元数qp相乘有

$$qp = q_r p_r - \vec{q} \cdot \vec{p} + q_r \vec{p} + \vec{q} p_r,$$

其中$\vec{q} \cdot \vec{p} = q_\mathrm{i} p_\mathrm{i} + q_\mathrm{j} p_\mathrm{j} + q_\mathrm{k} p_\mathrm{k}$, $\quad qp = \mathrm{i}(q_\mathrm{j} p_\mathrm{k} - q_\mathrm{k} p_\mathrm{j}) + \mathrm{j}(q_\mathrm{k} p_\mathrm{i} - q_\mathrm{i} p_\mathrm{k}) + \mathrm{k}(q_\mathrm{i} p_\mathrm{j} - q_\mathrm{j} p_\mathrm{i})$.

与复数情况类似，q的共轭四元数定义如下：

$$\bar{q} = q_r - \mathrm{i}q_\mathrm{i} - \mathrm{j}q_\mathrm{j} - \mathrm{k}q_\mathrm{k}. \tag{1}$$

注1 共轭函数颠倒了乘法的顺序，即$\overline{qp} = \bar{p}\bar{q}$.

由式(1)，四元数$q \in \mathbb{H}$的范数或模定义为：

$$|q| = \sqrt{q_r^2 + q_\mathrm{i}^2 + q_\mathrm{j}^2 + q_\mathrm{k}^2},$$

由此可见，$\bar{q}q = q\bar{q} = |q|^2$.

利用共轭和模，非零四元数$q \in \mathbb{H}$的逆为

$$q^{-1} = \frac{\bar{q}}{|q|^2}.$$

注2 当$|q| = 1$时，q为单位四元数，$q_r = 0$时，q称为纯四元数.

四元数q也可以表示为具有复数部分和虚数部分的复数：

$$q = z_1 + \mathrm{j}z_2,$$

其中$z_1 = q_r + \mathrm{i}q_\mathrm{i}$, $z_2 = q_\mathrm{j} + \mathrm{i}q_\mathrm{k}$. 这种表示法被称为Cayley-Dickson形式.

与复数情况类似,$\mathbb{H}$中函数f和g的内积定义如下：

$$\langle f, g\rangle_{L^2(\mathbb{H};\ \mathbb{H})} = \int_{\mathbb{H}} f(x)\cdot g(x)\mathrm{d}^4x,$$

其中$x\in\mathbb{H}$,$\mathrm{d}^4x = \mathrm{d}x_r\mathrm{d}x_i\mathrm{d}x_j\mathrm{d}x_k$. 每个四元数函数$f$都可以分解为

$$f(x) = f_r(x) + f_i(x)\mathrm{i} + f_j(x)\mathrm{j} + f_k(x)\mathrm{k},$$

其中$f_r(x)$,$f_i(x)$,$f_j(x)$和$f_k(x)$是x的实值函数.

特别地,对于$f = g$的情况,$L^2(\mathbb{H};\ \mathbb{H})$范数定义如下：

$$\| f \| = \left(\iint_{\mathbb{H}} |f(x)|^2\mathrm{d}^4x\right)^{\frac{1}{2}}.$$

1.2　四元数线性正则变换定义

定义 1　对于$f\in\mathbb{R}^2$,$\boldsymbol{M}_m = \begin{bmatrix} a_m & b_m \\ c_m & d_m\end{bmatrix}$,$a_m, b_m, c_m, d_m\in\mathbb{R}$,$m = 1,2$.矩阵参数满足$\det(\boldsymbol{M}_m) = 1$,二维右侧四元数线性正则变换的定义为：

$$\mathcal{L}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^2} f(\boldsymbol{x})K\mathrm{d}\boldsymbol{x},$$

其中$\boldsymbol{x} = (x_1, x_2)$,$\boldsymbol{\omega} = (\omega_1,\omega_2)\in\mathbb{R}^2$,$K = K_{\boldsymbol{M}_1}^{\mathrm{i}}(x_1,\omega_1)K_{\boldsymbol{M}_2}^{\mathrm{j}}(x_2,\omega_2)$,并且核函数($b_m\neq 0$)为：

$$K_{\boldsymbol{M}_1}^{\mathrm{i}}(x_1,\omega_1) = \frac{1}{\sqrt{2\pi|b_1|}}\exp\left[\mathrm{i}\left(\frac{a_1x_1^2}{2b_1} - \frac{x_1\omega_1}{b_1} + \frac{d_1}{2b_1}\omega_1^2\right)\right],$$

$$K_{\boldsymbol{M}_2}^{\mathrm{j}}(x_2,\omega_2) = \frac{1}{\sqrt{2\pi|b_2|}}\exp\left[\mathrm{j}\left(\frac{a_2x_2^2}{2b_2} - \frac{x_2\omega_2}{b_2} + \frac{d_2}{2b_2}\omega_2^2\right)\right].$$

注 3　本文只考虑$b_m\neq 0$的情况.

右侧四元数线性正则变换的逆变换定义为：

$$f(\boldsymbol{x}) = \int_{\mathbb{R}^2}\mathcal{L}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega})K_{\boldsymbol{M}_2^{-1}}^{\mathrm{j}}(x_2,\omega_2)K_{\boldsymbol{M}_1^{-1}}^{\mathrm{i}}(x_1,\omega_1)\mathrm{d}\boldsymbol{\omega},$$

其中$\boldsymbol{M}_m^{-1} = \begin{bmatrix} d_m & -b_m \\ -c_m & a_m\end{bmatrix}$,$m = 1,2$.

注 4　若$\boldsymbol{M}_m = \begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$,$m = 1,2$,则右侧四元数线性正则变换退化为右侧四元数傅里叶变换：

$$\mathcal{F}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(\boldsymbol{x})\exp(-\mathrm{i}x_1\omega_1)\exp(-\mathrm{j}x_2\omega_2)\mathrm{d}\boldsymbol{x}.$$

2　四元数线性正则变换

2.1　基于奇偶性的右侧四元数线性正则变换四个分量分解

首先我们可以展开右侧四元数线性正则变换的核

$$\begin{aligned} K &= K_{M_1}^{\mathrm{i}}(x_1,\omega_1)K_{M_2}^{\mathrm{j}}(x_2,\omega_2) = D_\gamma\exp(\mathrm{i}\gamma_1)\exp(\mathrm{j}\gamma_2) = D_\gamma(\cos\gamma_1 + \mathrm{i}\sin\gamma_1)(\cos\gamma_2 + \mathrm{j}\sin\gamma_2) \\ &= D_\gamma(\cos\gamma_1\cos\gamma_2 + \sin\gamma_1\cos\gamma_2\mathrm{i} + \cos\gamma_1\sin\gamma_2\mathrm{j} + \sin\gamma_1\cos\gamma_2\mathrm{k}),\end{aligned}$$

其中$D_\gamma = \dfrac{1}{2\pi\sqrt{|b_1b_2|}}$,$\gamma_m = \dfrac{a_mx_m^2}{2b_m} - \dfrac{x_m\omega_m}{b_m} + \dfrac{d_m}{2b_m}\omega_m^2$,$m = 1,2$.

因此右侧四元数线性正则变换可以分解为4个分量的和,即

$$\mathcal{L}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega}) = \mathcal{L}_{R,\mathrm{ee}}\{f\}(\boldsymbol{\omega}) + \mathcal{L}_{R,\mathrm{oe}}\{f\}(\boldsymbol{\omega})\mathrm{i} + \mathcal{L}_{R,\mathrm{eo}}\{f\}(\boldsymbol{\omega})\mathrm{j} + \mathcal{L}_{R,\mathrm{oo}}\{f\}(\boldsymbol{\omega})\mathrm{k}, \tag{2}$$

其中

$$\mathcal{L}_{R,\mathrm{ee}}\{f\}(\boldsymbol{\omega}) = D_\gamma\int_{\mathbb{R}^2} f_{\mathrm{ee}}(\boldsymbol{x})\cos\gamma_1\cos\gamma_2\mathrm{d}\boldsymbol{x},\qquad \mathcal{L}_{R,\mathrm{oe}}\{f\}(\boldsymbol{\omega}) = D_\gamma\int_{\mathbb{R}^2} f_{\mathrm{oe}}(\boldsymbol{x})\sin\gamma_1\cos\gamma_2\mathrm{d}\boldsymbol{x},$$

$$\mathcal{L}_{R,\mathrm{eo}}\{f\}(\boldsymbol{\omega})=D_\gamma\int_{\mathbb{R}^2}f_{\mathrm{eo}}(\boldsymbol{x})\cos\gamma_1\sin\gamma_2\mathrm{d}\boldsymbol{x},\qquad \mathcal{L}_{R,\mathrm{oo}}\{f\}(\boldsymbol{\omega})=D_\gamma\int_{\mathbb{R}^2}f_{\mathrm{oo}}(\boldsymbol{x})\sin_1\gamma_1\sin\gamma_2\mathrm{d}\boldsymbol{x}.$$

其中o, e表示奇偶.

2.2 右侧四元数线性正则变换的性质

定理1 设$f:\mathbb{R}^2\to\mathbb{H}$,且$\mathcal{L}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega})$表示$f$的右侧四元数线性正则变换,令$\mathcal{L}_R^{\mathbb{H}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})$, $\mathcal{L}_R^{\mathbb{H}}\left\{\frac{\partial f}{\partial x_2}\right\}(\boldsymbol{\omega})$分别表示$\frac{\partial f}{\partial x_1}$, $\frac{\partial f}{\partial x_2}$的右侧四元数线性正则变换,则有

$$\mathcal{L}_R^{\mathbb{H}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})=\frac{a_1}{b_1}\tilde{\mathcal{L}}_R^{\mathbb{H}}\{x_1f\}(\boldsymbol{\omega})(-\mathrm{i})-\frac{\omega_1}{b_1}\tilde{\mathcal{L}}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega})(-\mathrm{i}),\tag{3}$$

$$\mathcal{L}_R^{\mathbb{H}}\left\{\frac{\partial f}{\partial x_2}\right\}(\boldsymbol{\omega})=\frac{a_2}{b_2}\mathcal{L}_R^{\mathbb{H}}\{x_2f\}(\boldsymbol{\omega})(-\mathrm{j})-\frac{\omega_2}{b_2}\mathcal{L}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega})(-\mathrm{j}),\tag{4}$$

其中

$$\tilde{\mathcal{L}}_R^{\mathbb{H}}\{x_1f\}(\boldsymbol{\omega})=\mathcal{L}_{R,\mathrm{ee}}\{x_1f\}(\boldsymbol{\omega})-\mathcal{L}_{R,\mathrm{oe}}\{x_1f\}(\boldsymbol{\omega})\mathrm{i}-\mathcal{L}_{R,\mathrm{eo}}\{x_1f\}(\boldsymbol{\omega})\mathrm{j}-\mathcal{L}_{R,\mathrm{oo}}\{x_1f\}(\boldsymbol{\omega})\mathrm{k},$$
$$\tilde{\mathcal{L}}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega})=\mathcal{L}_{R,\mathrm{ee}}\{f\}(\boldsymbol{\omega})-\mathcal{L}_{R,\mathrm{oe}}\{f\}(\boldsymbol{\omega})\mathrm{i}-\mathcal{L}_{R,\mathrm{eo}}\{f\}(\boldsymbol{\omega})\mathrm{j}-\mathcal{L}_{R,\mathrm{oo}}\{f\}(\boldsymbol{\omega})\mathrm{k},$$
$$\mathcal{L}_R^{\mathbb{H}}\{x_2f\}(\boldsymbol{\omega})=\mathcal{L}_{R,\mathrm{ee}}\{x_2f\}(\boldsymbol{\omega})+\mathcal{L}_{R,\mathrm{oe}}\{x_2f\}(\boldsymbol{\omega})\mathrm{i}+\mathcal{L}_{R,\mathrm{eo}}\{x_2f\}(\boldsymbol{\omega})\mathrm{j}+\mathcal{L}_{R,\mathrm{oo}}\{x_2f\}(\boldsymbol{\omega})\mathrm{k},$$
$$\mathcal{L}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega})=\mathcal{L}_{R,\mathrm{ee}}\{f\}(\boldsymbol{\omega})+\mathcal{L}_{R,\mathrm{oe}}\{f\}(\boldsymbol{\omega})\mathrm{i}+\mathcal{L}_{R,\mathrm{eo}}\{f\}(\boldsymbol{\omega})\mathrm{j}+\mathcal{L}_{R,\mathrm{oo}}\{f\}(\boldsymbol{\omega})\mathrm{k}.$$

证明 本文只证式(3). 由式(2)可得

$$\mathcal{L}_R^{\mathbb{H}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})=\mathcal{L}_{R,\mathrm{ee}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})+\mathcal{L}_{R,\mathrm{oe}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})\mathrm{i}+\mathcal{L}_{R,\mathrm{eo}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})\mathrm{j}+\mathcal{L}_{R,\mathrm{oo}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})\mathrm{k},$$

其中

$$\mathcal{L}_{R,\mathrm{ee}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})=D_\gamma\int_{\mathbb{R}^2}\frac{\partial f}{\partial x_1}\cos\gamma_1\cos\gamma_2\mathrm{d}\boldsymbol{x},\qquad \mathcal{L}_{R,\mathrm{oe}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})=D_\gamma\int_{\mathbb{R}^2}\frac{\partial f}{\partial x_1}\sin\gamma_1\cos\gamma_2\mathrm{d}\boldsymbol{x},$$

$$\mathcal{L}_{R,\mathrm{eo}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})=D_\gamma\int_{\mathbb{R}^2}\frac{\partial f}{\partial x_1}\cos\gamma_1\sin\gamma_2\mathrm{d}\boldsymbol{x},\qquad \mathcal{L}_{R,\mathrm{oo}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})=D_\gamma\int_{\mathbb{R}^2}\frac{\partial f}{\partial x_1}\sin_1\gamma_1\sin\gamma_2\mathrm{d}\boldsymbol{x}.$$

$$\gamma_m=\frac{a_mx_m^2}{2b_m}-\frac{x_m\omega_m}{b_m}+\frac{d_m}{2b_m}\omega_m^2,\ m=1,2.$$

对于每一个可积光滑函数f和每一个$x_1,x_2\in\mathbb{R}$,有$\lim\limits_{x_1\to\infty}f(x)=0$,因此可得

$$\mathcal{L}_{R,\mathrm{ee}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})=\frac{a_1}{b_1}\mathcal{L}_{R,\mathrm{oe}}\{x_1f\}(\boldsymbol{\omega})-\frac{w_1}{b_1}\mathcal{L}_{R,\mathrm{oe}}\{f\}(\boldsymbol{\omega}),\ \mathcal{L}_{R,\mathrm{oe}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})=-\left(\frac{a_1}{b_1}\mathcal{L}_{R,\mathrm{ee}}\{x_1f\}(\boldsymbol{\omega})-\frac{w_1}{b_1}\mathcal{L}_{R,\mathrm{ee}}\{f\}(\boldsymbol{\omega})\right),$$

$$\mathcal{L}_{R,\mathrm{eo}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})=\frac{a_1}{b_1}\mathcal{L}_{R,\mathrm{oo}}\{x_1f\}(\boldsymbol{\omega})-\frac{w_1}{b_1}\mathcal{L}_{R,\mathrm{oo}}\{f\}(\boldsymbol{\omega}),\ \mathcal{L}_{R,\mathrm{oo}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})=-\left(\frac{a_1}{b_1}\mathcal{L}_{R,\mathrm{oo}}\{x_1f\}(\boldsymbol{\omega})-\frac{w_1}{b_1}\mathcal{L}_{R,\mathrm{oo}}\{f\}(\boldsymbol{\omega})\right).$$

所以

$$\begin{aligned}\mathcal{L}_R^{\mathbb{H}}\left\{\frac{\partial f}{\partial x_1}\right\}(\boldsymbol{\omega})&=\frac{a_1}{b_1}\left[\mathcal{L}_{R,\mathrm{oe}}\{x_1f\}(\boldsymbol{\omega})-\mathcal{L}_{R,\mathrm{ee}}\{x_1f\}(\boldsymbol{\omega})\mathrm{i}+\mathcal{L}_{R,\mathrm{oo}}\{x_1f\}(\boldsymbol{\omega})\mathrm{j}-\mathcal{L}_{R,\mathrm{eo}}\{x_1f\}(\boldsymbol{\omega})\mathrm{k}\right]\\&\quad-\frac{w_1}{b_1}\left[\mathcal{L}_{R,\mathrm{oe}}\{f\}(\boldsymbol{\omega})-\mathcal{L}_{R,\mathrm{ee}}\{f\}(\boldsymbol{\omega})\mathrm{i}+\mathcal{L}_{R,\mathrm{oo}}\{f\}(\boldsymbol{\omega})\mathrm{j}-\mathcal{L}_{R,\mathrm{eo}}\{f\}(\boldsymbol{\omega})\mathrm{k}\right]\\&=\frac{a_1}{b_1}\left[\mathcal{L}_{R,\mathrm{ee}}\{x_1f\}(\boldsymbol{\omega})-\mathcal{L}_{R,\mathrm{oe}}\{x_1f\}(\boldsymbol{\omega})\mathrm{i}-\mathcal{L}_{R,\mathrm{eo}}\{x_1f\}(\boldsymbol{\omega})\mathrm{j}-\mathcal{L}_{R,\mathrm{oo}}\{x_1f\}(\boldsymbol{\omega})\mathrm{k}\right](-\mathrm{i})\\&\quad-\frac{w_1}{b_1}\left[\mathcal{L}_{R,\mathrm{ee}}\{f\}(\boldsymbol{\omega})-\mathcal{L}_{R,\mathrm{oe}}\{f\}(\boldsymbol{\omega})\mathrm{i}-\mathcal{L}_{R,\mathrm{eo}}\{f\}(\boldsymbol{\omega})\mathrm{j}-\mathcal{L}_{R,\mathrm{oo}}\{f\}(\boldsymbol{\omega})\mathrm{k}\right](-\mathrm{i}).\end{aligned}$$

此证结束,同理可证式(4).

推论1 设$f:\mathbb{R}^2\to\mathbb{H}$, 且$\mathcal{L}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega})$表示$f$的右侧四元数线性正则变换,令$\mathcal{L}_R^{\mathbb{H}}\left\{\frac{\partial^2 f}{\partial x_1^2}\right\}(\boldsymbol{\omega})$, $\mathcal{L}_R^{\mathbb{H}}\left\{\frac{\partial^2 f}{\partial x_1\partial x_2}\right\}(\boldsymbol{\omega})$和$\mathcal{L}_R^{\mathbb{H}}\left\{\frac{\partial^2 f}{\partial x_2^2}\right\}(\boldsymbol{\omega})$, 分别表示$\frac{\partial^2 f}{\partial x_1^2}$, $\frac{\partial^2 f}{\partial x_1\partial x_2}$和$\frac{\partial^2 f}{\partial x_2^2}$的右侧四元数线性正则变换,

则有

$$\mathcal{L}_R^{\mathbb{H}}\left\{\frac{\partial^2 f}{\partial x_1^2}\right\}(\boldsymbol{\omega}) = \frac{a_1^2}{b_1^2}\tilde{\mathcal{L}}_R^{\mathbb{H}}\{x_1^2 f\}(\boldsymbol{\omega}) - \frac{a_1\omega_1}{b_1^2}\tilde{\mathcal{L}}_R^{\mathbb{H}}\{x_1 f\}(\boldsymbol{\omega}) - \frac{\omega_1^2}{b_1^2}\tilde{\mathcal{L}}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega}),$$

$$\mathcal{L}_R^{\mathbb{H}}\left\{\frac{\partial^2 f}{\partial x_1 \partial x_2}\right\}(\boldsymbol{\omega}) = -\frac{a_1 a_2}{b_1 b_2}\tilde{\mathcal{L}}_R^{\mathbb{H}}\{x_1 x_2 f\}(\boldsymbol{\omega}) + \frac{a_1\omega_2}{b_1 b_2}\tilde{\mathcal{L}}_R^{\mathbb{H}}\{x_1 f\}(\boldsymbol{\omega}) + \frac{\omega_1 a_2}{b_1 b_2}\tilde{\mathcal{L}}_R^{\mathbb{H}}\{x_2 f\}(\boldsymbol{\omega}) - \frac{\omega_1\omega_2}{b_1 b_2}\tilde{\mathcal{L}}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega}),$$

$$\mathcal{L}_R^{\mathbb{H}}\left\{\frac{\partial^2 f}{\partial x_2^2}\right\}(\boldsymbol{\omega}) = \frac{a_2^2}{b_2^2}\mathcal{L}_R^{\mathbb{H}}\{x_2^2 f\}(\boldsymbol{\omega}) - \frac{a_2\omega_2}{b_2^2}\mathcal{L}_R^{\mathbb{H}}\{x_2 f\}(\boldsymbol{\omega}) + \frac{\omega_2^2}{b_2^2}\tilde{\mathcal{L}}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega}).$$

其中

$$\tilde{\mathcal{L}}_R^{\mathbb{H}}\{x_1^2 f\}(\boldsymbol{\omega}) = \mathcal{L}_{R,\mathrm{ee}}\{x_1^2 f\}(\boldsymbol{\omega}) - \mathcal{L}_{R,\mathrm{oe}}\{x_1^2 f\}(\boldsymbol{\omega})\mathrm{i} - \mathcal{L}_{R,\mathrm{eo}}\{x_1^2 f\}(\boldsymbol{\omega})\mathrm{j} - \mathcal{L}_{R,\mathrm{oo}}\{x_1^2 f\}(\boldsymbol{\omega})\mathrm{k},$$

$$\mathcal{L}_R^{\mathbb{H}}\{x_2^2 f\}(\boldsymbol{\omega}) = \mathcal{L}_{R,\mathrm{ee}}\{x_2^2 f\}(\boldsymbol{\omega}) + \mathcal{L}_{R,\mathrm{oe}}\{x_2^2 f\}(\boldsymbol{\omega})\mathrm{i} + \mathcal{L}_{R,\mathrm{eo}}\{x_2^2 f\}(\boldsymbol{\omega})\mathrm{j} + \mathcal{L}_{R,\mathrm{oo}}\{x_2^2 f\}(\boldsymbol{\omega})\mathrm{k},$$

$$\tilde{\mathcal{L}}_R^{\mathbb{H}}\{x_1 x_2 f\}(\boldsymbol{\omega}) = \mathcal{L}_{R,\mathrm{ee}}\{x_1 x_2 f\}(\boldsymbol{\omega}) - \mathcal{L}_{R,\mathrm{oe}}\{x_1 x_2 f\}(\boldsymbol{\omega})\mathrm{i} - \mathcal{L}_{R,\mathrm{eo}}\{x_1 x_2 f\}(\boldsymbol{\omega})\mathrm{j} - \mathcal{L}_{R,\mathrm{oo}}\{x_1 x_2 f\}(\boldsymbol{\omega})\mathrm{k}.$$

3　例子及仿真

在本节中，通过1个高斯信号示例对所提出的右侧四元数线性正则变换进行了演示．

例1　我们考虑以下四元数值信号：

$$f(\boldsymbol{x}) = g(\boldsymbol{x})\exp\left[-\left(\frac{\alpha}{2}x_1^2 + \frac{\beta}{2}x_2^2\right)\right] = (1 + 3\mathrm{i} + 5\mathrm{j} + 7\mathrm{k})\exp\left[-\left(\frac{\alpha}{2}x_1^2 + \frac{\beta}{2}x_2^2\right)\right]. \tag{5}$$

其中$\alpha, \beta \in \mathbb{R}^+$. 我们绘制了四元数值信号，四元数傅里叶变换以及四元数线性正则变换$f(\boldsymbol{x})$的各分量，如图1~3所示．

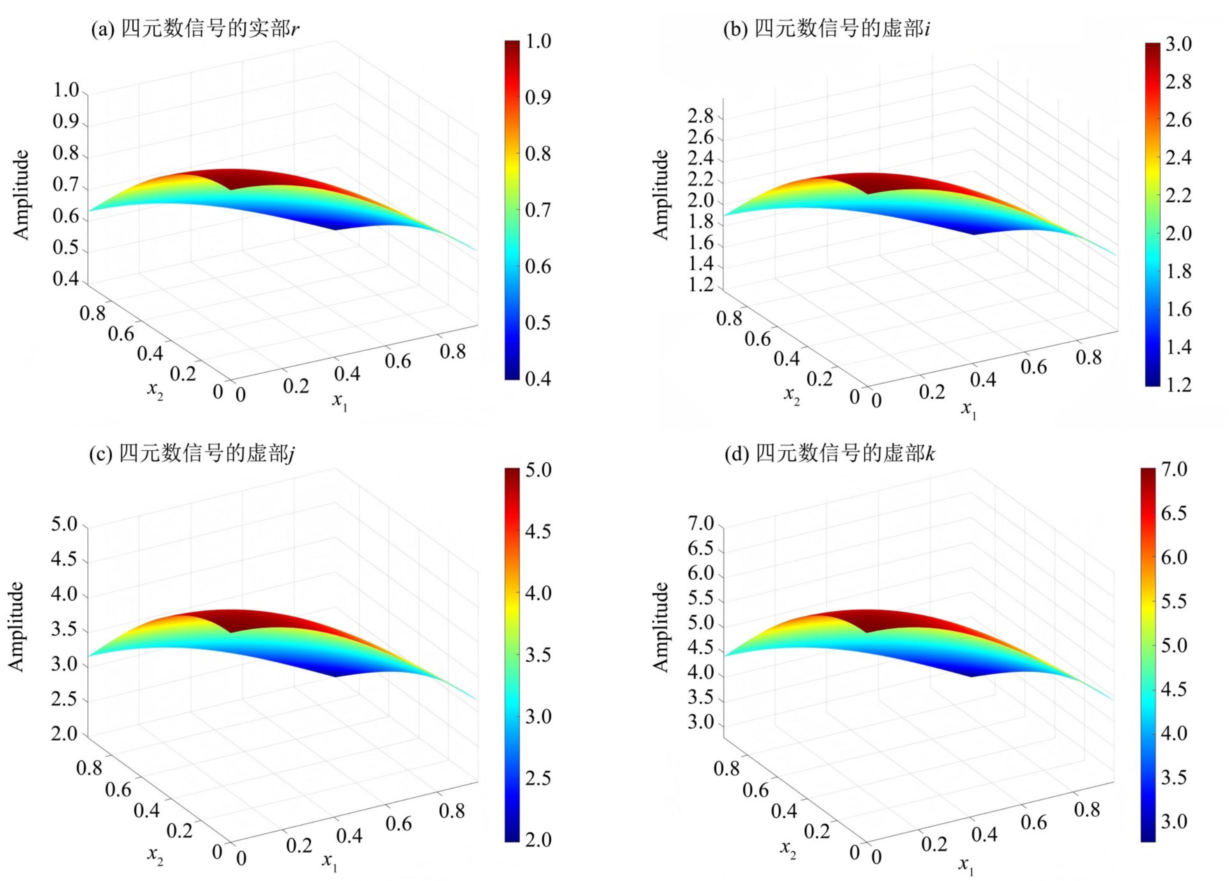

图1　四元数信号各分量的空间域分布

Fig.1　The distribution of each component of the quaternion signal in spatial domain

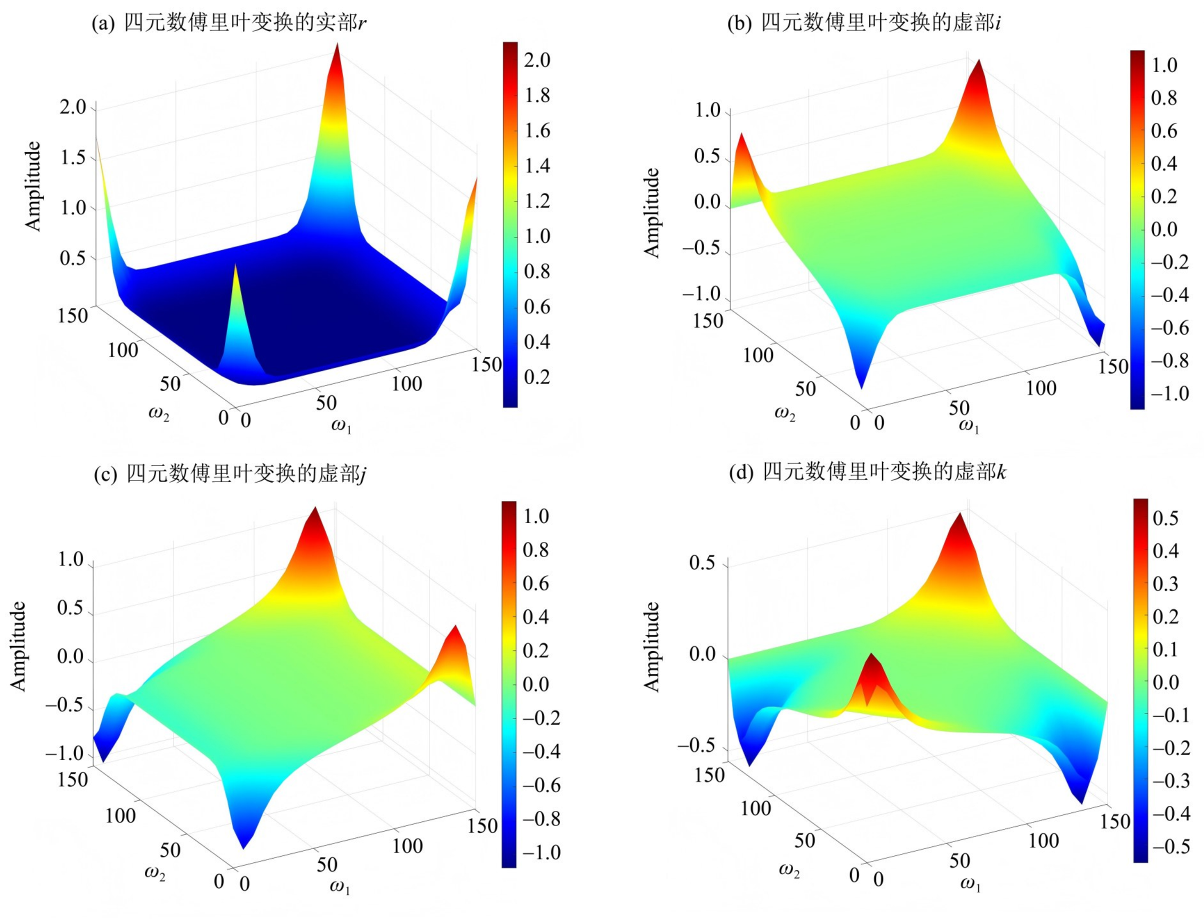

图2 四元数傅里叶变换的值分布

Fig.2 The value distribution of quaternion Fourier transform

接下来，我们按如下方式计算函数$f(\boldsymbol{x})$的右侧四元数线性正则变换：

$$
\begin{aligned}
\mathcal{L}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega}) &= \int_{\mathbb{R}^2} f(\boldsymbol{x}) K \mathrm{d}\boldsymbol{x} \\
&= D_\gamma \int_{\mathbb{R}^2} g(\boldsymbol{x}) \exp\left[\mathrm{i}\left(\frac{a_1 x_1^2}{2b_1} - \frac{x_1\omega_1}{b_1} + \frac{d_1}{2b_1}\omega_1^2\right)\right] \exp\left[\mathrm{j}\left(\frac{a_2 x_2^2}{2b_2} - \frac{x_2\omega_2}{b_2} + \frac{d_2}{2b_2}\omega_2^2\right)\right] \exp\left[-\left(\frac{\alpha}{2}x_1^2 + \frac{\beta}{2}x_2^2\right)\right] \mathrm{d}\boldsymbol{x} \\
&= D_\gamma g(\boldsymbol{x}) \int_{\mathbb{R}^2} \exp\left(\frac{\mathrm{i}a_1 x_1^2}{2b_1} - \frac{\alpha}{2}x_1^2 + \frac{\mathrm{i}x_1\omega_1}{b_1} - \frac{\mathrm{i}d_1\omega_1^2}{2b_1}\right) \exp\left(-\frac{\mathrm{j}a_2 x_2^2}{2b_2} - \frac{\beta}{2}x_2^2 - \frac{\mathrm{j}x_2\omega_2}{b_2} + \frac{\mathrm{j}d_2\omega_2^2}{2b_2}\right) \mathrm{d}\boldsymbol{x} \\
&= D_\gamma g(\boldsymbol{x}) \exp\left(-\frac{\mathrm{i}d_1\omega_1^2}{2b_1}\right) \exp\left(\frac{\mathrm{j}d_2\omega_2^2}{2b_2}\right) \int_{\mathbb{R}^2} \exp\left[-x_1^2\left(\frac{\mathrm{i}a_1}{2b_1} - \frac{\alpha}{2}\right) - \frac{\mathrm{i}x_1\omega_1}{b_1}\right] \exp\left[-x_2^2\left(\frac{\mathrm{j}a_2}{2b_2} + \frac{\beta}{2}\right) - \frac{\mathrm{j}x_2\omega_2}{b_2}\right] \mathrm{d}\boldsymbol{x} \\
&= g(\boldsymbol{x}) \exp\left(-\frac{\mathrm{i}d_1\omega_1^2}{2b_1}\right) \exp\left(\frac{\mathrm{j}d_2\omega_2^2}{2b_2}\right) \exp\left(\frac{\omega_1^2}{2b_1(\alpha b_1 - \mathrm{i}a_1)}\right) \frac{1}{\sqrt{\mathrm{i}a_1 - \alpha b_1}} \exp\left(-\frac{\omega_2^2}{2b_2(\beta b_2 + \mathrm{j}a_2)}\right) \frac{1}{\sqrt{\beta b_2 + \mathrm{j}a_2}} \\
&= g(\boldsymbol{x}) \frac{1}{\sqrt{\mathrm{i}a_1 - \alpha b_1}\sqrt{\beta b_2 + \mathrm{j}a_2}} \exp\left(\frac{\mathrm{i}d_1\omega_1^2}{2b_1}\right) \exp\left(-\frac{\mathrm{j}d_2\omega_2^2}{2b_2}\right) \exp\left(-\frac{\omega_1^2}{2b_1(\alpha b_1 - \mathrm{i}a_1)}\right) \exp\left(-\frac{\omega_2^2}{2b_2(\beta b_2 + \mathrm{j}a_2)}\right) \\
&= g(\boldsymbol{x}) \frac{1}{\sqrt{\mathrm{i}a_1 - \alpha b_1}\sqrt{\beta b_2 + \mathrm{j}a_2}} \exp\left[\frac{\omega_1^2}{2b_1}\left(\mathrm{i}d_1 - \frac{1}{\alpha b_1 - \mathrm{i}a_1}\right)\right] \exp\left[\frac{\omega_2^2}{2b_2}\left(\mathrm{j}d_2 - \frac{1}{\beta b_2 + \mathrm{j}a_2}\right)\right].
\end{aligned}
$$

令$A_\alpha = \dfrac{1}{\mathrm{i}a_1 - \alpha b_1}$，$B_\beta = \dfrac{1}{\beta b_2 + \mathrm{j}a_2}$，有

$$
\mathcal{L}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega}) = g(\boldsymbol{x}) \sqrt{A_\alpha B_\beta} \exp\left[\frac{\omega_1^2}{2b_1}(\mathrm{i}d_1 + A_\alpha)\right] \exp\left[\frac{\omega_2^2}{2b_2}(\mathrm{j}d_2 - B_\beta)\right].
$$

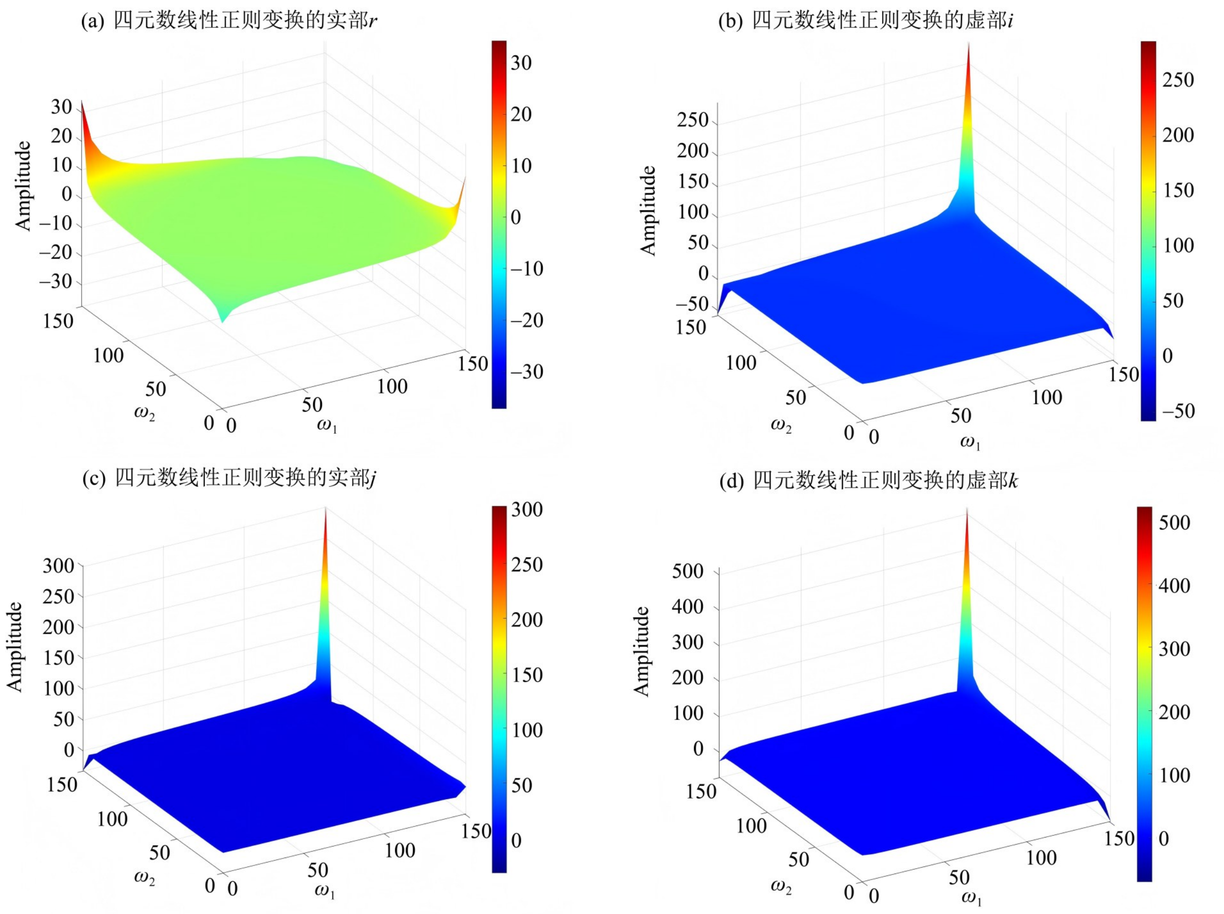

图 3　四元数线性正则变换的值分布

Fig.3　The value distribution of quaternion linear canonical transform

令 $a_m = 1, b_m = 1, c_m = -1, d_m = 0\,(m = 1, 2), \alpha = \beta = 1$,可得 QLCT:

$$\mathcal{L}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega}) = (1 + 3\mathrm{i} + 5\mathrm{j} + 7\mathrm{k})\sqrt{\frac{-1 - \mathrm{i} + \mathrm{j} + \mathrm{k}}{4}}\exp\left(-\frac{\omega_1^2}{4b_1}\right)\exp\left(-\mathrm{i}\frac{\omega_1^2}{4b_1}\right)\exp\left(\frac{\omega_2^2}{4b_2}\right)\exp\left(-\mathrm{j}\frac{\omega_2^2}{4b_2}\right)$$

$$= \sqrt{\frac{-1 - \mathrm{i} + \mathrm{j} + \mathrm{k}}{4}}\exp\left(-\frac{\omega_1^2}{4b_1}\right)\exp\left(\frac{\omega_2^2}{4b_2}\right)(1 - 3\mathrm{i} - 5\mathrm{j} - 7\mathrm{k})\exp\left(\mathrm{i}\frac{\omega_1^2}{4b_1}\right)\exp\left(-\mathrm{j}\frac{\omega_2^2}{4b_2}\right).$$

令 $\theta_1 = \frac{\omega_1^2}{4b_1}, \theta_2 = \frac{\omega_2^2}{4b_2}$, 有

$$\begin{aligned}
\mathcal{L}_R^{\mathbb{H}}\{f\}(\boldsymbol{\omega}) &= \sqrt{\frac{-1 - \mathrm{i} + \mathrm{j} + \mathrm{k}}{4}}\exp(-\theta_1 + \theta_2)(1 - 3\mathrm{i} - 5\mathrm{j} - 7\mathrm{k})\exp(\mathrm{i}\theta_1)\exp(-\mathrm{j}\theta_2)\\
&= E(\theta)(1 - 3\mathrm{i} - 5\mathrm{j} - 7\mathrm{k})(\cos\theta_1 + \mathrm{i}\sin\theta_1)(\cos\theta_2 - \mathrm{j}\sin\theta_2)\\
&= E(\theta)(1 - 3\mathrm{i} - 5\mathrm{j} - 7\mathrm{k})(\cos\theta_1\cos\theta_2 + \mathrm{i}\sin\theta_1\cos\theta_2 - \mathrm{j}\cos\theta_1\sin\theta_2 - \mathrm{k}\sin\theta_1\sin\theta_2)\\
&= E(\theta)(\cos\theta_1\cos\theta_2 + 3\sin\theta_1\cos\theta_2 - 5\cos\theta_1\sin\theta_2 - 7\sin\theta_1\sin\theta_2)\\
&\quad + E(\theta)\mathrm{i}(-3\cos\theta_1\cos\theta_2 + \sin\theta_1\cos\theta_2 - 7\cos\theta_1\sin\theta_2 + 5\sin\theta_1\sin\theta_2)\\
&\quad + E(\theta)\mathrm{j}(-5\cos\theta_1\cos\theta_2 - 7\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 - 3\sin\theta_1\sin\theta_2)\\
&\quad + E(\theta)\mathrm{k}(-7\cos\theta_1\cos\theta_2 + 5\sin\theta_1\cos\theta_2 + 3\cos\theta_1\sin\theta_2 - \sin\theta_1\sin\theta_2),
\end{aligned}$$

其中 $E(\theta) = \sqrt{\frac{-1 + \mathrm{i} - \mathrm{j} + \mathrm{k}}{4}}\exp(-\theta_1 + \theta_2)$.

4　分析与讨论

本节针对第 3 节中高斯四元数信号的仿真结果(图 1~3)展开分析,通过对比原始信号、四元数傅里叶变

换及四元数线性正则变换的图像特征，进行具体分析与比较.

从图1呈现的原始四元数高斯信号空间域分布来看，其各个分量均展现出典型的高斯特性，即中心区域振幅最高，向边缘逐渐衰减，这与理论预期相符，验证了所构造信号的合理性. 实部r的振幅范围相对较小，分布也更为平缓，而虚部i, j, k的振幅范围依次增大，这正是由于函数$f(\boldsymbol{x})$的3个虚部的系数不同而造成的，这一现象体现了四元数中标量部分与矢量部分在空间域能量分布上的差异，虚部k分量（图1d）峰值达7左右，显著高于实部（峰值约1）、虚部i（峰值约3）、虚部j（峰值约5）分量，可对应式（5）所提出的高斯函数，印证四元数矢量部分常表征空间物理量强度的特性，且各分量高斯分布中心一致，为后续变换分析提供统一基准，契合四元数矢量部分能量主导的特性.

图2所示的四元数傅里叶变换结果，作为四元数线性正则变换在特定参数下的退化形式，呈现出频率域的对称分布特征，中心区域振幅形成峰值并向高频区域对称衰减，这与傅里叶变换将空间域高斯信号映射为频率域高斯分布的理论特性一致. 不过，相较于空间域，其振幅范围整体缩小，表明傅里叶变换对信号能量进行了重新分配，高频成分的能量受到抑制. 同时，虚部分量出现明显的正负交替，这是由于四元数乘法的非交换性导致的，体现了四元数傅里叶变换与复数傅里叶变换在相位特性上的差异.

图3展示的四元数线性正则变换结果，与四元数傅里叶变换相比呈现出显著不同的特征，这主要源于其核函数受矩阵参数调控的灵活性. 从振幅范围来看，线性正则变换的虚部分量振幅明显大于傅里叶变换，甚至超过原始信号，说明通过调整参数，它能够有效放大特定频率成分的能量，克服了傅里叶变换参数固定带来的局限性. 这种能量放大特性对于提取信号中的微弱特征具有重要意义，也体现了线性正则变换在信号表征上的优势.

综合来看，四元数线性正则变换通过参数的灵活调控，在四元数信号处理中展现出比四元数傅里叶变换更强的适应性. 它不仅能够验证理论推导的正确性，还能根据实际需求放大关键频率成分，保留信号本质特征，这使其在需要灵活处理四元数信号的领域具有潜在的应用价值. 而四元数傅里叶变换作为其特例，虽然在对称信号的全局频率分析中仍有作用，但在复杂信号的精细处理上则稍显不足.

5 结 论

本文系统研究了四元数线性正则变换，先介绍了四元代数的基本理论和四元数线性正则变换的定义，接着完成了基于奇偶性的右侧四元数线性正则变换分量分解，推导证明了其相关性质及推论，最后通过高斯信号实例进行仿真演示，验证了理论的有效性.

本方法为四元数信号的特征提取（如三维运动图像、电磁矢量信号处理）提供了新工具，未来可结合深度学习框架优化参数自适应选择.

参考文献：

BHAT M Y, DAR A H, 2022. Convolution and correlation theorems for Wigner-Ville distribution associated with the quaternion offset linear canonical transform[J]. Signal Image Video Process, 16(5): 1235-1242.

BRACEWELL R N, 1989. The Fourier transform[J]. Sci Am, 260(6): 86-95.

GAO W B, LI B Z, 2020. Quaternion windowed linear canonical transform of two-dimensional signals[J]. Adv Appl Clifford Algebr, 30(1): 16.

HAMILTON W R, 1866. Elements of quaternions[M]. London: Longmans, Green and Co.

HEALY J J, KUTAY M A, OZAKTAS H M, et al, 2016. Linear canonical transforms: Theory and applications[M]. New York: Springer.

HU X, CHENG D, KOU K I, 2023. Convolution theorems associated with quaternion linear canonical transform and applications[J]. Signal Process, 202: 108743.

JIANG N, FENG Q, YANG X, et al, 2025. The octonion linear canonical transform: Properties and applications[J]. Chaos Solitons Fractals, 192: 116039.

LI M, YUAN X, CHEN H, et al, 2020. Quaternion discrete Fourier transform-based color image watermarking method using quaternion QR decomposition[J]. IEEE Access，8：72308−72315.

LIU C，HAN P，KANG J，2025. Integrating short−time linear canonical transform and joint space-time-frequency analysis for advanced representation of subsurface information in ground penetrating radar[J]. Sci Rep，15：23315.

MAJORKOWSKA-MECH D，CARIOW A，2023. One-dimensional quaternion discrete Fourier transform and an approach to its fast computation[J]. Electronics，12(24)：4974.

MEI Y Y，FENG Q，GAO X X，et al，2023. Convolution theorem associated with the QWFRFT[J]. Chin J Electron，32(3)：485−492.

SHAH F A，TANTARY A Y，2022. Multi-dimensional linear canonical transform with applications to sampling and multiplicative filtering[J]. Multidimens Syst Signal Process，33(2)：621−650.

SIDDIQUI S，SAMAD M A，ISMOILJONOVICH F D，2024. One dimensional quaternion linear canonical transform in probability theory[J]. Signal Image Video Process，18(12)：9419−9430.

WANG R B，FENG Q，2024a. Weighted convolution for quaternion linear canonical cosine transform and its application[J]. Axioms，13(6)：402.

WANG R B，FENG Q，2024b. Fractional mixed weighted convolution and its application in convolution integral equations[J]. J Math，2024：5375401.

WANG R B，FENG Q，JI J Y，2024c. The discrete convolution for fractional cosine-sine series and its application in convolution equations[J]. AIMS Math，9(2)：2641−2656.

XIANG Y，YUAN S，FENG Q，2023. Fractional Fourier cosine and sine Laplace weighted convolution and its application[J]. IET Signal Process，17(2)：e12170.

YANG H H，FENG Q，WANG X X，et al，2024. Reduced biquaternion windowed linear canonical transform：Properties and applications[J]. Mathematics，12(5)：743.

YANG X，FENG Q，JIANG N，et al，2025. Properties and applications of octonion fractional Fourier transform for 3-D octonion signals[J]. Digit Signal Process，165：105339.

ZHU X，ZHENG S，2021. Uncertainty principles for the two-sided offset quaternion linear canonical transform[J]. Math Methods App Sciences，44(18)：14236−14255.

（责任编辑　冯兆永）